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On Soluble *n*th Order Linear Differential Equations

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The expansion of a product of n linear factors in the operator D is obtained and this leads to a simple solution of the corresponding nth order linear differential equations with variable coefficients.

INTRODUCTION

The determination of the solution of the differential equation

$$[(x^3 - 1)\,D^3 - 3ax^2D^2 + 3a(a+1)\,xD - a(a+1)(a+2)]\,y = 0 \tag{1}$$

was proposed by Harley in problem 2491 [1]. There were two published solutions; one by Kitchin and the other by the proposer. The former involved some rather "tricky" operational manipulation reducing the problem to linear first order equations. It was also noted, without any details,[1] that one could similarly solve an analogous nth order equation which was explicitly given. The proposer's solution was based on factoring the differential operator in (1) into three linear factors in D. In this note, we justify the operational solution and thus also show how it applies to more general linear differential equations. Then we give an nth order extention of the proposer's solution which also includes the nth order extension of Kitchin as a special case. Here, due to the resultant symmetry of the constants occurring in the equation, a solution simpler than that of the proposer is obtained. Finally, we treat some exceptional cases which were not considered by Harley and Kitchin.

OPERATIONAL SOLUTION

Kitchin lets $x = e^t$, to transform (1) into

$$e^{3t}(D-a)(D-a-1)(D-a-2)\,y = D(D-1)(D-2)\,y, \tag{2}$$

[1] Perhaps the details were eliminated by the editor.

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where now D means d/dt. Equation (2) is then rewritten as

$$y - \frac{D-a-3}{D} e^t \cdot \frac{D-a-3}{D} e^t \cdot \frac{D-a-3}{D} e^t y = 0$$

or

$$\left\{1 - \left(\frac{D-a-3}{D} e^t\right)^3\right\} y = 0.$$

Thus,

$$\omega\, Dy = (D-a-3)\, e^t y,$$

where $\omega^3 = 1$. Then solving

$$y = C_1(t-1)^{a+2} + C_2(t-\omega)^{a+2} + C_3(t-\omega)^{a+2}.$$

In order to justify the above operational method, we consider the more general equation

$$L(D-2P)\, L(D-P)\, L(D)\, y = M(D)\, e^Q M(D)\, e^Q M(D)\, e^Q y. \tag{3}$$

where $L(D)$ and $M(D)$ are linear operators (polynomials) in D and P and Q are functions of t such that

$$Q(t) = \int^t P(t)\, dt.$$

Kitchin's solution is based on the exponential shift theorem

$$e^Q F(D) \equiv F(D-P) e^Q \qquad (F\text{—a polynomial in } D). \tag{4}$$

From (3), we formally have

$$\begin{aligned} L(D-P)\, L(D) y &= \frac{M(D)}{L(D-2P)} e^Q M(D)\, e^Q M(D)\, e^Q y \\ &= M(D)\, e^Q \frac{M(D)}{L(D-P)} e^Q M(D)\, e^Q y \\ &= M(D)\, e^Q M(D)\, e^Q \frac{M(D)}{L(D)} e^Q y. \end{aligned}$$

Then similarly,

$$L(D) y = M(D)\, e^Q \frac{M(D)}{L(D)} e^Q \frac{M(D)}{L(D)} e^Q y$$

and

$$y = \left\{ \frac{M(D)}{L(D)} e^Q \right\}^3 y. \tag{5}$$

It then follows that the solution of (5) is given by the solution of

$$\omega L(D) y = M(D) e^Q y \qquad (\omega^3 = 1). \tag{6}$$

We now justify these latter manipulations by showing that the solutions of (6) do indeed satisfy (3) provided that $M(D)$ commutes with both $L(D-P)$ and $L(D-2P)$.

Multiplying (6) by e^Q and using (4), we get

$$\omega L(D-P) e^Q y = e^Q M(D) e^Q y$$

and then

$$\omega M(D) L(D-P) e^Q y = M(D) e^Q M(D) e^Q y.$$

Assuming $M(D)$ commutes with $L(D-P)$, we then obtain

$$\omega L(D-P) M(D) e^Q y = \omega^2 L(D-P) L(D) y = M(D) e^Q M(D) e^Q y.$$

Operating on the latter equation with $M(D) e^Q$ and using (4) again, yields

$$\omega^2 M(D) L(D-2P) L(D-P) e^Q y = \{M(D) e^Q\}^3 y.$$

Assuming that $M(D)$ also commutes with $L(D-2P)$, we finally obtain (3), i.e.,

$$\begin{aligned}\omega^2 L(D-2P) L(D-P) M(D) e^Q y &= L(D-2P) L(D-P) L(D) y \\ &= \{M(D) e^Q\}^3 y.\end{aligned}$$

Similarly, the solution of

$$L(D-(n-1P) L(D-(n-2)P) \cdots L(D) y = \{M(D) e^Q\}^n y \tag{7}$$

is obtainable from that of

$$\omega L(D) y = M(D) e^Q y \qquad (\omega^n = 1),$$

provided that $M(D)$ commutes with $L(D - rP)$, $r = 1, 2, \ldots, n-1$.

Kitchin also notes that the differential equation in x corresponding to

$$\left\{ 1 - \left(\frac{D-a-n}{D} e^t \right)^n \right\} y = 0 \tag{8}$$

is given by

$$x^n \frac{d^n y}{dx^n} - \binom{n}{1} a_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + (-1)^n \binom{n}{n} a_n = \frac{d^n y}{dx^n}, \tag{9}$$

where $a_r = a(a+1)\cdots(a+r-1)$. To verify this, note that (8) can be rewritten as

$$D(D-1)\cdots(D-n+1)y = [(D-a-n)e^t]^n y.$$

Now, using $x = e^t$, we obtain

$$x^n D^n y = [(xD-a-n)x]^n y,$$

so we have to show that

$$\begin{aligned} x^{-n}[(xD-a-n)x]^n y &= x^{-n}[x^2D - x(a+n-1)]^n y \\ &= \text{left hand side of (9)}. \end{aligned}$$

This immediately follows from an identity of Berkowič and Kvalwasser [2], i.e.,

$$[x^2D + ax]^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(a+n)}{\Gamma(a+n-k)} x^{2n-k} D^{n-k}. \tag{10}$$

A simpler derivation and generalizations of (10) are given in [3]. Also, (10) can be obtained using Gauss's theorem for hypergeometric series [4].

Factorization Solution

The proposer solved (1) by noting it could be rewritten in the factored form

$$[(x-1)D-a][(x-\omega^2)D-a-1][(x-\omega)D-a-2]y=0 \tag{11}$$

which was then integrated successively to find the general solution. Equation (1) can be generalized to

$$[(x+a_1)D-\lambda][(x+a_2)D-\lambda-1]\cdots[(x+a_n)D-\lambda-n+1]y=0. \tag{12}$$

We shall show that the expanded form of (12) is symmetric in all the constants a_k ($k = 1, 2, \ldots, n$). Consequently, all the solutions are given by those of

$$[(x+a_k)D-\lambda-n+1]y_k=0$$

or

$$y_k = C_k(x + a_k)^{\lambda + n - 1} \qquad (k = 1, 2, \ldots, n) \tag{13}$$

and finally

$$y = y_1 + y_2 + \cdots + y_n.^2 \tag{14}$$

We now show that the expanded form of (12) is given by

$$[T_n D^n - \lambda_1 T_{n-1} D^{n-1} + \cdots + (-1)^n \lambda_n T_0] y = 0, \tag{15}$$

where $\lambda_r = \lambda(\lambda + 1) \cdots (\lambda + r - 1)$ and the T_r's are the elementary symmetric functions of the $(x + a_r)$'s, i.e.,

$$(t + x + a_1)(t + x + a_2) \cdots (t + x + a_n) \equiv T_0 t^n + T_1 t^{n-1} + \cdots + T_n.$$

Our proof is by induction. Obviously $(12) \cong (15)$ for $n = 1$. Assume this is also valid for $n = k$. We then have to show that

$$\begin{aligned}[(x + a_0) D - \lambda + 1][T_k D^k - \lambda_1 T_{k-1} D^{k-1} + \cdots + (-1)^k \lambda_k T_0] \\ = [\bar{T}_{k+1} D^{k+1} - (\lambda - 1) \bar{T}_k D^k + \cdots + (-1)^{k-1} (\lambda - 1)_{k+1} \bar{T}_0],\end{aligned}$$

where

$$\bar{T}_{k+1} = (x + a_0) T_k, \qquad \bar{T}_r = T_r + (x + a_0) T_{r-1}$$
$$(r = 1, 2, \ldots, k), \qquad \bar{T}_0 = T_0 = 1.$$

The rest follows since

$$(x + a_0) D\{\lambda_{k-r} T_r D^r\} = \lambda_{k-r}\{(x + a_0) T_r D^{r+1} + (x + a_0) T'_r D^r\}$$

and $T'_r = (k - r) T_{r-1}$.

It is to be noted that the solution (14) is not complete if $\lambda = -1, -2, \ldots, -(n-1)$ since then the y_k of (13) are not linearly independent. For example, if $\lambda = 1 - n$, $y_k =$ constant. For $\lambda = 0$, the y_k are just linearly independent and we could just as well write the solution in the form

$$y = k_0 + k_1 x + \cdots + k_{n-1} x^{n-1}.$$

To pick up the additional linearly independent solutions other than those given by (13), we go back to (12). Assume now that $\lambda = -k$, where $1 \leqslant k \leqslant n - 1$. Equation (12) now becomes

$$\begin{aligned}[(x + a_1) D + k] \cdots [(x + a_k) D + 1] \\ \times [(x + a_{k+1}) D + 0] \cdots [(x + a_n) D + k - n + 1] y = 0.\end{aligned}$$

[2] We are assuming the a_r's are all distinct. If not, the missing solutions can be obtained by usual limiting procedures.

Then by virtue of (15),

$$[(x+a_{k+1})D+0]\cdots[(x+a_n)D+k-n+1]y$$
$$=(x+a_{k+1})(x+a_{k+2})\cdots(x+a_n)D^{n-k}y\equiv u.$$

We now solve

$$[(x+a_1)D+k][(x+a_2)D+k-1]\cdots[(x+a_k)D+1]u=0,$$

again using (13) with $n=k$, $\lambda=-k$, giving

$$u=(x+a_{k+1})(x+a_{k+2})\cdots(x+a_n)D^{n-k}y=\sum_{j=1}^{k}C_j(x+a_j)^{-1}.$$

Whence,

$$y=\sum_{j=1}^{k}C_j\int\cdots\int\frac{(dx)^{n-k}}{(x+a_j)(x+a_{k+1})\cdots(x+a_n)}$$
$$+C_{k+1}+C_{k+2}+\cdots+C_nx^{n-k-1}. \qquad (16)$$

This solution can be expressed symmetrically in terms of the a_i's by writing it as

$$y=\int\cdots\int\frac{C_1'+C_2'x+\cdots+C_k'x^{k-1}}{(x+a_1)(x+a_2)\cdots(x+a_n)}(dx)^{n-k}+\sum_{r=1}^{n-k}C_{k+r}x^{r-1}. \qquad (16')$$

In particular for the case $k=n-1$, we get

$$y=C_0+\sum_{j=1}^{n}C_j\log(x+a_j),$$

where $C_1+C_2+\cdots+C_n=0$.

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